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Analytic continuation of the single-eigenvalue probability density function

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Abstract. An exact expression for the single-eigenvalue probability density function for any dimension of a Gaussian orthogonal ensemble is found by analytic continuation. It is shown that the form can be used to derive a number of exact results in a simple way. In odd dimensions it is shown that the single-eigenvalue probability density function can be written as a sum of two densities.

1. Introduction

The concept of a random matrix was originally introduced by Wigner (1957) to study the statistical properties of the compound-nucleus level widths and their positions. It was soon established that if each matrix element of a real symmetric Hamiltonian (Mehta 1967) has an independent Gaussian distribution such that the dispersion of the off-diagonal elements is half the dispersion of the diagonal element then the joint distribution of the eigenvalues is a Wishart distribution. It was shown by Dyson (1962) that Gaussian ensembles can be classified as orthogonal, unitary or symplectic depending how they behave under rotations and time-reversal operation, e.g., if a system is invariant both under time reversal and rotation then the appropriate matrix ensemble for such a system is a Gaussian orthogonal ensemble (GOE). In further development of matrix ensemble theory one was interested in finding an expression for the probability density function of the single eigenvalue by integrating out all but one eigenvalue in the joint-eigenvalue distribution. It was found that mathematically it is easier to work with the Gaussian unitary ensemble (GUE) than with GOE. The mathematical difficulty in GOE arises because of a factor which is a product of the absolute value of the differences of the eigenvalues. The method of integration over alternate variables was developed by Mehta (1967) to overcome this difficulty. Even during the early development of GOE it was found that GOE has another feature which distinguishes it from GUE, namely the probability density function of the single eigenvalue had two different mathematical forms depending on whether the dimension of the matrix N was even or odd. An exact expression for $N=2m$ was subsequently derived by Mehta and Gaudin (1960) for the probability density function of the single eigenvalue using the method of integrating over alternate variables. The purpose of the present work is to show that one could carry out further mathematical steps so that the expression can be analytically continued for odd values of N also. In the process we find a simpler exact expression for the probability density of the single eigenvalue for any N.

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In § 2 we describe the formulation and in § 3 present some exact results. Concluding remarks are presented in § 4.

2. Formulation

Let us consider the joint distribution of N real eigenvalues $E_1, E_2, \ldots, E_N$. It is given by (Mehta 1967):

$$P(\{E_i\}) = K \exp\left(-\tfrac{1}{2} \sum_{\mu=1}^{N} E_\mu^2\right) \prod_{\mu<\nu} |E_\mu - E_\nu| \tag{1}$$

where K is the normalisation constant.

Taking $N = 2m$, the method of integrating over alternate variables (Mehta and Gaudin 1960) gives the following probability density function of the single eigenvalue:

$$P_{2m}(E) = \frac{1}{2m} \sum_{i=0}^{2m-2} \phi_i^2(E) + \frac{[2(2m-1)]^{1/2}}{4m} \phi_{2m-1}(E) \int_0^E \phi_{2m-2}(X)\,\mathrm{d}X \tag{2}$$

where $\phi_i(X)$ are the normalised harmonic oscillator wavefunctions given by

$$\phi_i(X) = (\sqrt{\pi} 2^i i!)^{-1/2} \exp(-\tfrac{1}{2}X^2) H_i(X). \tag{3}$$

It is convenient to rewrite expression (2) using the following relation satisfied by the harmonic oscillator wavefunctions:

$$\sqrt{2}\phi'_{2m-1} = (2m-1)^{1/2}\phi_{2m-2} - (2m)^{1/2}\phi_{2m}. \tag{4}$$

The probability density $P_{2m}(E)$ can then be written as

$$P_{2m}(E) = \frac{1}{2m} \sum_{i=0}^{2m-1} \phi_i^2(E) + \frac{1}{2\sqrt{m}} \phi_{2m-1} \int_0^E \phi_{2m}(X)\,\mathrm{d}X. \tag{5}$$

We note here that in GUE, the second factor in expression (5) is absent. It is this second factor which has to be recast to find the analytically continued form of the probability density function for any dimension N.

Let I denote the integral

$$I = \int_0^E \phi_{2m}(X)\,\mathrm{d}X. \tag{6}$$

Using the definition of the Hermite polynomial in terms of a confluent hypergeometric function (Abramowitz and Stegun 1965) and carrying out the integration over X, we can write I as

$$I = \frac{(-1)^m \pi^{1/4}[(2m)!]^{1/2}}{2^{m+1} m!} E \exp(-\tfrac{1}{2}E^2) \sum_{r=0}^{\infty} \frac{\Gamma(-m+r)}{\Gamma(-m)} \frac{1}{\Gamma(\frac{3}{2}+r)} \frac{1}{r!} M(1, \tfrac{3}{2}+r, \tfrac{1}{2}E^2) \tag{7}$$

where $M(a, b, x)$ denotes a confluent hypergeometric function (Abramowitz and Stegun 1965).

Expanding the confluent hypergeometric function and writing the powers of E in terms of Hermite polynomials using the relation

$$X^{2\mu+1} = \frac{(2\mu+1)!}{2^{2\mu+1}} \sum_{r=0}^{\infty} \frac{H_{2r+1}(x)}{(2r+1)!(\mu-r)!} \tag{8}$$

we find after some simplification that I is given by

$$I=\sum_{r=m}^{\infty}\left(\frac{\Gamma(m+\frac{1}{2})r!}{m!\Gamma(r+\frac{3}{2})}\right)^{1/2}\phi_{2r+1}(E). \tag{9}$$

Expression (9) can be rewritten as

$$I=\left(\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\right)^{1/2}\sum_{\nu=0}^{\infty}\left(\frac{\Gamma(m+\nu+1)}{\Gamma(m+\nu+\frac{3}{2})}\right)^{1/2}\phi_{2m+2\nu+1}(E). \tag{10}$$

Using expressions (5) and (10) it is easy to see that the analytically continued form of the single-eigenvalue probability density function $P_N(E)$ can be written as

$$P_N(E)=\frac{1}{N}\sum_{i=0}^{N-1}\phi_i^2(E)+\frac{1}{N}\left(\frac{\Gamma[(N+1)/2]}{\Gamma(N/2)}\right)^{1/2}\phi_{N-1}(E)\sum_{\nu=0}^{\infty}\left(\frac{\Gamma(\nu+1+\frac{1}{2}N)}{\Gamma(\nu+\frac{3}{2}+\frac{1}{2}N)}\right)^{1/2}\phi_{2\nu+N+1}. \tag{11}$$

An alternative form of $P_N(E)$ can also be obtained using expressions (5) and (9) and the generating function for the Hermite polynomials. It is given by

$$P_N=\frac{1}{N}\sum_{i=0}^{N-1}\phi_i^2(E)+\frac{1}{\sqrt{\pi}2^N}\exp(-E^2)H_{N-1}(E)\frac{1}{2\pi \mathrm{i}}\oint\frac{\mathrm{d}\xi}{\xi^{N+2}}\times\exp(-\xi^2+2E\xi)\,{}_2F_0(\tfrac{1}{2}(N+2),1;\xi^{-2}) \tag{12}$$

where ${}_2F_0$ is the generalised hypergeometric function (Abramowitz and Stegun 1965).

3. Some exact results using the analytically continued form

In this section we shall give some exact results which can be derived using expressions (11) and (12). Some of the results which are familiar from earlier work provide a check on these expressions, while others are given for the first time here.

One nice feature of both expressions (11) and (12) is that because of the orthogonality of the ϕ_i, it is almost trivial to see that $P_N(E)$ is normalised to unity. This feature was not there in $P_{2m}(E)$ given by expression (2) which had an integral over ϕ_{2m-2}. The same is true about a later form (Mehta 1971, Mehta and Pandey 1983) of the single-eigenvalue probability density function in which the lower limit of the integral was extended to $-\infty$ to include both even and odd dimensions.

It is also easy to calculate low-order moments of the single eigenvalue using expression (11), e.g., the second moment can be calculated using the relation

$$X^2\phi_n(X)=\tfrac{1}{2}[(n+1)(n+2)]^{1/2}\phi_{n+2}+\tfrac{1}{2}(2n+1)\phi_n+\tfrac{1}{2}[n(n-1)]^{1/2}\phi_{n-2} \tag{13}$$

and the orthogonality of the ϕ_n, and is given by

$$\langle E^2\rangle=\tfrac{1}{4}(N+1). \tag{14}$$

This checks with the value calculated using the relation

$$\sum_{i=1}^{N}E_i^2=\mathrm{Tr}\,H^2 \tag{15}$$

and using the Gaussian distribution of the matrix elements of H.

We next show the rederivation of the two-dimensional distribution of the single eigenvalue using expression (11). Using the following sum rule (Hansen 1975):

$$\sum_{k=0}^{\infty} \frac{H_{2k+1}}{2^{2k}\Gamma(k+\frac{3}{2})} = \frac{2}{\sqrt{\pi}} EM(1, \tfrac{3}{2}, \tfrac{1}{2}E^2) \tag{16}$$

in expression (11) we find that

$$P_2(E) = \frac{1}{2\sqrt{\pi}} \exp(-E^2) + \frac{\sqrt{2\pi}}{2} E \exp\left(-\frac{E^2}{2}\right) \operatorname{erf}\left(\frac{E}{\sqrt{2}}\right) \tag{17}$$

where erf is the error function (Abramowitz and Stegun 1965), which is the same expression as one finds by integrating out one of the eigenvalues from a joint distribution of two eigenvalues.

We next show that the probability density function of the single eigenvalue can be written as a sum of two densities when the dimension N is odd.

Putting $N = 2m+1$ in expression (11) and using the relation

$$\exp\left(\frac{X^2}{2}\right) = \sum_{n=0}^{\infty} \frac{H_{2n}(X)}{2^{2n-1/2}\Gamma(n+1)} \tag{18}$$

which can be easily proved using the generating function for the Hermite polynomials, we can write P_{2m+1} as

$$P_{2m+1} = \frac{1}{2m+1} \sum_{i=0}^{2m} \phi_i^2 + \frac{1}{2^{2m+3/2}\Gamma(m+\frac{3}{2})} \times \exp(-E^2) H_{2m} \left[\exp\left(\frac{E^2}{2}\right) - \sqrt{2} \sum_{\mu=0}^{m} \frac{H_{2\mu}}{2^{2\mu}\mu!}\right]. \tag{19}$$

Defining a new set of harmonic oscillator wavefunctions

$$\chi_\mu(E) = 2^{-1/4}\phi_\mu\left(\frac{E}{\sqrt{2}}\right) \tag{20}$$

and writing the second term in expression (19) in terms of $\chi_\mu(E)$ we can write P_{2m+1} as

$$P_{2m+1}(E) = \frac{1}{2m+1} \sum_{i=0}^{2m} \phi_i^2(E) - \sum_{r=0}^{m} \left(\frac{m!\Gamma(r+\frac{1}{2})}{\Gamma(m+\frac{1}{2})r!}\right)^{1/2} \phi_{2m}(E)\phi_{2r}(E) + \frac{1}{2m+1} \sum_{\mu=0}^{m} 2^{\mu} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\mu)\mu!}\right)^{1/2} \frac{m!}{(m-\mu)!} \chi_{2\mu}(E)\chi_0(E). \tag{21}$$

Thus in odd dimensions the probability density function of the single eigenvalue can be expressed as a sum of two densities, one given in terms of ϕ_μ and the other in terms of χ_μ.

As a further application of the form of the single-eigenvalue probability density $P_N(E)$ given by expression (11) we now calculate the exact Fourier transform of $P_N(E)$. We write the Fourier transform $g(\alpha)$ as

$$g(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}E \exp(\mathrm{i}\alpha E) P_N(E). \tag{22}$$

Using expression (11) and the generating function (Abramowitz and Stegun 1965) for Hermite polynomials, we find that it is exactly given by

$$g(\alpha)=\frac{1}{\sqrt{2\pi}}\frac{1}{N}\exp\left(-\frac{\alpha^2}{4}\right)L_{N-1}^{(1)}(\alpha^2/2)-\frac{1}{2\sqrt{2\pi}}\frac{1}{N}\left(\frac{\Gamma[(N+1)/2](N-1)!}{\Gamma(N/2)}\right)^{1/2}$$
$$\times\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\alpha^{2\nu+2}}{2^\nu}\left(\frac{\Gamma(\nu+1+N/2)}{\Gamma(\nu+\frac{3}{2}+N/2)(2\nu+N+1)!}\right)^{1/2}$$
$$\times\frac{1}{[(2\nu+2)!]^2}L_{N-1}^{(2\nu+2)}(\alpha^2/2) \tag{23}$$

where $L_{N-1}^{(r)}$ is the associated Laguerre polynomial.

We see from expression (23) that first part of the Fourier transform is exactly the same as one finds in GUE (Ullah 1985).

From expression (23) we can derive the corrections applied to the semicircular distribution which arises from the second part. For this purpose we use Hilb's type of formula for $L_{N-1}^{(r)}$. It is given by (Bateman 1953):

$$\exp(-\tfrac{1}{2}x)x^{\alpha/2}L_n^{(\alpha)}(x)=\frac{\Gamma(n+\alpha+1)}{(\nu/4)^{\alpha/2}n!}J_\alpha((\nu x)^{1/2})+\mathrm{O}(n^{\alpha/2-1/4}) \tag{24}$$

where $\alpha>-1$, $0<x\leqslant\omega<\infty$ and $\nu=4n+2\alpha+2$.

Retaining the first term of the summation in expression (23) and using the above asymptotic expression we can write $g(\alpha)$ as

$$g(\alpha)=\frac{2}{\sqrt{2\pi}}\frac{J_1(\sqrt{2N}\alpha)}{(\sqrt{2N}\alpha)}-\frac{1}{4\sqrt{2\pi}}\frac{1}{(N+\frac{1}{2})}J_2(\sqrt{2N+1}\alpha)\ldots. \tag{25}$$

Taking the inverse Fourier transform (Abramowitz and Stegun 1965) we find that $P_N(E)$ for large N can be written as

$$P_N(E)=\frac{2}{\pi}\frac{1}{2N}\left(1-\frac{E^2}{2N}\right)^{1/2}+\frac{1}{2(2N+1)^{3/2}}\frac{T_2(\omega)}{(1-\omega^2)^{1/2}} \tag{26}$$

where $\omega=E/(2N+1)$ and $T_2(\omega)=(2\omega^2-1)$ is the Chebyshev polynomial of the second kind. $P_N(E)$ is zero for $E^2/2N$ or $\omega^2>1$.

Lastly we remark that one could derive a closed form expression for all the even moments M_{2n} of the single eigenvalue using the exact Fourier transform given by expression (23). It is given by

$$M_{2n}=\frac{(2n)!}{4^n}\left[F(-(N-1),-n;2;2)+\frac{1}{N}\sum_{\nu=0}^{n-1}\frac{1}{(2\nu+2)!}\frac{2^{\nu+1}}{(n-\nu-1)!}\right.$$
$$\times F(-(N-1),1-n+\nu;2\nu+3;2)$$
$$\left.\times\left(\frac{(2\nu+N+1)!}{(N-1)!}\frac{\Gamma[(N+1)/2]}{\Gamma(N/2)}\frac{\Gamma(\nu+1+N/2)}{\Gamma(\nu+\frac{3}{2}+N/2)}\right)^{1/2}\right]. \tag{27}$$

Again the first part of M_{2n} is the same as one finds in GUE and gives the asymptotic form of M_{2n}.

4. Concluding remarks

We have shown that starting from the probability density function of the single eigenvalue for even dimension one can analytically continue the expression so that it is valid both for even and odd dimensions. The new expression has the nice feature that low-order moments of the single eigenvalue can be easily written down; the normalisation integral in particular is trivial to find. A new interpretation of the probability density function in odd dimensions is given. It can be written as a sum of two densities.

Further it is shown that using the exact Fourier transform of the probability density function, one can write a closed form expression for all the even moments of the single eigenvalue. The Fourier transform can also be used to find corrections applied to a semicircular distribution.

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